Problem 2.3

Consider the function

$$\psi(z,t) = \frac{A}{(z-vt)^2+1}$$

where A is a constant. Show that it is a solution of the differential wave equation. Determine the speed of the wave and the direction of propagation.

Solution

The aim is to show that the given function satisfies the wave equation.

$$\frac{\partial^2\psi}{\partial t^2} \stackrel{?}{=} v^2\frac{\partial^2\psi}{\partial z^2}$$

Start taking derivatives, using the chain rule where appropriate.

$$\frac{\partial\psi}{\partial t} = \frac{\partial}{\partial t}\left[\frac{A}{(z-vt)^2+1}\right] = -\frac{A}{[(z-vt)^2+1]^2}\cdot 2(z-vt)\cdot(-v) = 2Av\frac{z-vt}{[(z-vt)^2+1]^2}$$

$$\begin{aligned}
\frac{\partial^2\psi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\psi}{\partial t}\right) = \frac{\partial}{\partial t}\left\{2Av\frac{z-vt}{[(z-vt)^2+1]^2}\right\} \\
&= 2Av\frac{\left[\frac{\partial}{\partial t}(z-vt)\right][(z-vt)^2+1]^2 - \left\{\frac{\partial}{\partial t}[(z-vt)^2+1]^2\right\}(z-vt)}{[(z-vt)^2+1]^4} \\
&= 2Av\frac{(-v)[(z-vt)^2+1]^2 - \left\{-4v(z-vt)[(z-vt)^2+1]\right\}(z-vt)}{[(z-vt)^2+1]^4} \\
&= 2Av\frac{v[(z-vt)^2+1][3(z-vt)^2-1]}{[(z-vt)^2+1]^4} \\
&= 2Av^2\frac{3(z-vt)^2-1}{[(z-vt)^2+1]^3}
\end{aligned}$$

$$\frac{\partial\psi}{\partial z} = \frac{\partial}{\partial z}\left\{\frac{A}{(z-vt)^2+1}\right\} = -\frac{A}{[(z-vt)^2+1]^2}\cdot 2(z-vt)\cdot(1) = -2A\frac{z-vt}{[(z-vt)^2+1]^2}$$

$$\begin{aligned}
\frac{\partial^2\psi}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right) = \frac{\partial}{\partial z}\left\{-2A\frac{z-vt}{[(z-vt)^2+1]^2}\right\} \\
&= -2A\frac{\left[\frac{\partial}{\partial z}(z-vt)\right][(z-vt)^2+1]^2 - \left\{\frac{\partial}{\partial z}[(z-vt)^2+1]^2\right\}(z-vt)}{[(z-vt)^2+1]^4} \\
&= -2A\frac{(1)[(z-vt)^2+1]^2 - \left\{4(z-vt)[(z-vt)^2+1]\right\}(z-vt)}{[(z-vt)^2+1]^4} \\
&= 2A\frac{[(z-vt)^2+1][3(z-vt)^2-1]}{[(z-vt)^2+1]^4} \\
&= 2A\frac{3(z-vt)^2-1}{[(z-vt)^2+1]^3}
\end{aligned}$$

Therefore,

$$\frac{\partial^2\psi}{\partial t^2} = 2Av^2\frac{3(z-vt)^2-1}{[(z-vt)^2+1]^3} = v^2\left\{2A\frac{3(z-vt)^2-1}{[(z-vt)^2+1]^3}\right\} = v^2\frac{\partial^2\psi}{\partial z^2},$$

which means the given function for $\psi(z,t)$ is a solution of the wave equation.

$\psi(z,t) = A/[(z-vt)^2+1]$ travels in the positive z-direction if v is positive, and it travels in the negative z-direction if v is negative. The wave travels at speed $|v|$.